A NOT SO SHORT INTRODUCTION TO GROTHENDIECK TOPOI (EXTENDED ABSTRACT)

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Contents

1. Introduction

In this thesis we introduce the notion of a Grothendieck topos, and describe two classes of topoi associated to a topological space, the *Zariski* topos and the *étale* topos. We show that these two topoi are equivalent. This allows us to infer upon the structure we lose when we disregard the structure sheaf $\mathscr{O}_X$ of a scheme X, where the two topoi are different.
Finally we define *geometric morphisms* and give an explicit description of the category of points of the Zariski topos $\mathrm{Shv}(X)$.

2. Preliminaries

Definition. Given a category $\boldsymbol{\mathcal{C}}$, a functor $F : \boldsymbol{\mathcal{C}}^{\mathrm{op}} \to \boldsymbol{Set}$ is called a *presheaf* (or a presheaf of sets) on $\boldsymbol{\mathcal{C}}$.

Let $\boldsymbol{\mathcal{C}}$ be a locally small category, then consider the class of contravariant functors for each $c \in \boldsymbol{\mathcal{C}}$, $h_c := \mathrm{Mor}_{\boldsymbol{\mathcal{C}}}(-, c)$ sending c' to $\mathrm{Mor}_{\boldsymbol{\mathcal{C}}}(c', c)$. These functors are called *representable presheaves*.
Now we present a standard lemma in category theory, the proof can be found in any standard text such as [Lan98] or [Moe92].

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Lemma 1. *Given a locally small category $\boldsymbol{\mathcal{C}}$ and a functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{Set}$, there is a natural bijection:*

$$\gamma_{F,c} : Nat(Mor_{\boldsymbol{\mathcal{C}}}(c, -), F) \cong F(c)$$

Note that a representable presheaf on $\boldsymbol{\mathcal{C}}$, $\mathrm{Mor}_{\boldsymbol{\mathcal{C}}}(-, c)$ is the same as a representable functor $\mathrm{Mor}_{\boldsymbol{\mathcal{C}}^{\mathrm{op}}}(c, -)$ and so we have the following version of the Yoneda lemma for contravariant functors:

Corollary 2. *Let $\boldsymbol{\mathcal{C}}$ be a locally small category and $F : \boldsymbol{\mathcal{C}}^{op} \to \boldsymbol{Set}$, then there is a natural bijection:*

$$Nat(Mor_{\boldsymbol{\mathcal{C}}}(-, c), F) \cong F(c)$$

It is now important to note that a monomorphism in a category $\boldsymbol{\mathcal{C}}$ can be thought of as a limit or, more precisely, as a pullback, for a proof see, for example [Moe92]:

Lemma 3. *Let $\boldsymbol{\mathcal{C}}$ be a category and $f : A \to B$ a morphism in $\boldsymbol{\mathcal{C}}$. Then f is a monomorphism if and only if the following diagram is a pullback square:*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd} \tag{1}$$

Theorem 4. *Any category $\boldsymbol{\mathcal{D}}$ with arbitrary products and equalisers of pairs of arrows is complete.*

Proof. (sketch) Simply note that the equaliser on the middle row in the diagram below provides a limit for F.

$$\begin{tikzcd} & F(d) \arrow[r, "1_{F(d)}"] & F(d) \\ L \arrow[r, "e"] & \prod_{c \in \boldsymbol{\mathcal{C}}} F(c) \arrow[u, "\pi_d"] \arrow[d, "\pi_c"] \arrow[r, shift left, "q"] \arrow[r, shift right, "p"'] & \prod_{f:c \to d \text{ in } \boldsymbol{\mathcal{C}}} F(d)_f \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\ & F(c) \arrow[r, "F(f)"] & F(d) \end{tikzcd} \tag{2}$$

□

Note that in the case of $\boldsymbol{Set}$ limits have a simple model given by $\mathrm{Lim}\, F \cong \mathrm{Mor}_{\boldsymbol{Set}}(\{*\}, F)$.

Definition. A functor F is said to be *continuous* if it commutes with all limits, i.e. if $F(\mathrm{Lim}\, G) = \mathrm{Lim}\, (F \circ G)$

And Theorem4 reduces continuity to commuting with products and equalisers.

Definition. Given a pair of functors L, R with reversed domain and codomain as below:

$$\boldsymbol{\mathcal{C}} \underset{R}{\overset{L}{\rightleftarrows}} \boldsymbol{\mathcal{D}}$$

we say L is a *left adjoint* to R if there exists a natural bijection:

$$\psi_{a,b} : \mathrm{Mor}_{\boldsymbol{\mathcal{D}}}(La, b) \cong \mathrm{Mor}_{\boldsymbol{\mathcal{C}}}(a, Rb)$$

Equivalently an *adjunction* can be given by a pair of natural transformations $\eta : \mathsf{Id}_{\boldsymbol{\mathcal{C}}} \to RL$ and $\varepsilon : LR \to \mathsf{Id}_{\boldsymbol{\mathcal{D}}}$ so that the *triangular identities* below are satisfied

$$
\begin{tikzcd}
R \arrow[r, "\eta_R"] \arrow[dr, equal] & RLR \arrow[d, "R(\varepsilon)"] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "L(\eta)"] \arrow[dr, equal] & LRL \arrow[d, "\varepsilon_L"] \\
& L
\end{tikzcd}
$$

For the proof of the equivalence of the formulations see [dC13]

Definition. A pair of adjoint functors $L \dashv R$ define an equivalence of categories if the unit and counit are natural isomorphisms.

Lemma 5. *Every left (resp. right) adjoint between cocomplete (resp. complete) categories is cocontinuous (resp. continuous).*

2.1. **Kan Extensions.** Now we focus on functor categories. In this section let $i : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{C}}$ be the inclusion functor of a subcategory of $\boldsymbol{\mathcal{C}}$.

Definition. Let $T : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{D}}$ be a functor, where $\boldsymbol{\mathcal{D}}$ is complete. The right Kan extension of T, is the functor $i_K(T) : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ given on objects by:

$$i_K(T)(c) = \operatorname*{Lim}_{f \in (c \downarrow \boldsymbol{\mathcal{M}})} T \circ Q(f)$$

where Q is the projection $(c \downarrow \boldsymbol{\mathcal{M}}) \to \boldsymbol{\mathcal{M}}$. If $\boldsymbol{\mathcal{D}}$ is cocomplete we can define the left Kan extension (on objects) as:

$$i^K(T)(c) = \operatorname*{Colim}_{f \in (\boldsymbol{\mathcal{M}} \downarrow c)} T \circ Q(f)$$

where Q is the projection $(\boldsymbol{\mathcal{M}} \downarrow c) \to \boldsymbol{\mathcal{M}}$.

Proposition 6. *The right and left Kan extensions provide a right and left adjoint to the functor i^* : $Func(\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{D}}) \to Func(\boldsymbol{\mathcal{M}}, \boldsymbol{\mathcal{D}})$ given by precomposition with the inclusion i.*

Proof. The proof can again be found in the standard texts, such as [Lan98]. $\square$

2.2. **Filtered Categories and Colimits.** Filtered categories share properties with filtered posets. In this section we present the definition of filtered category and prove an important theorem which states that filtered colimits commute with finite limits to $\boldsymbol{Set}$.

Definition. A category I is said to be *filtered* if it satisfies:

(1) For any objects $j, j' \in I$ there exists an object $k \in I$ and morphisms $j \to k$ and $j' \to k$.
(2) For any object $i \in I$ and any pair of parallel arrows $f, g : i \rightrightarrows j$ there exists an object k and an arrow $w : j \to k$ so that $w \circ f = w \circ g$

Note that every category with a terminal object is filtered.
We call colimits from a filtered category, *filtered colimits*.

Theorem 7. *Filtered colimits commute with finite limits in $\boldsymbol{Set}$.*

We say a functor is *left exact* if it commutes with finite limits.

Proof. The proof can be seen in [Lan98] $\square$

2.3. **Irreducible Closed Sets.**

Definition. A topological space, T is said to be *irreducible* if it cannot be written as the union of two (and hence, any finite number of) proper closed subsets.
We say that T is an *irreducible closed* subset of X if it is a closed subset of X, and is an irreducible space when endowed with the subspace topology.

The irreducible closed subsets $T \subseteq X$ of a topological space form a partially ordered set (or poset) under inclusion, this then becomes a category to which we call $\operatorname{Irr}(X)$.

2.4. Étale Maps and Étale Spaces.

Definition. Given two topological spaces Y and X we say that a continuous map $p : Y \to X$ is *étale*, or a *local homeomorphism*, if for every $y \in Y$ there exist open neighborhoods $V \subseteq Y$ of y and $U \subseteq X$ of $p(y)$ so that $p|_V$ is a homeomorphism between V and U.

Given a map $p : Y \to X$ we call the set $p^{-1}(x)$ for $x \in X$, the *fiber over* x, and we call a space Y with an étale map $p : Y \to X$, an *étale space* over X.
Note that the fiber of an étale map over any point $x \in X$ must be a discrete set and an étale map is always an open map.
It's also easy to show that the composition of étale maps is an étale map, and so:

Theorem 8. *The class of topological spaces with étale maps between them, form a subcategory of* $\boldsymbol{Top}$*, which we call* $\textbf{Ét}$.

Now define the category of étale spaces over a fixed space X as the category $\textbf{Ét}(X) := (\textbf{Ét} \downarrow X)$. We can easily check that if p and q are étale maps to X and f is such that $f \circ p = q$ then f is also étale, and so:

Proposition 9. *Given any topological space X, the category $\textbf{Ét}(X)$ is a full subcategory of $\boldsymbol{Top}/X$.*

2.5. Presheaves and Sheaves on a Topological Space.

Definition. Given a topological space X, we define a presheaf on X to be a functor $F : O(X)^{\mathrm{op}} \to \boldsymbol{Set}$, where $O(X)$ is the poset category of open sets. We write $\mathrm{PSh}(X)$ for the category of presheaves on X with natural transformations as morphisms.

We call the elements of $F(U)$ the *sections* of F on U, and we denote the restriction of a section $s \in F(U)$ to $F(V)$ where $V \subseteq U$ by $s|_V$. Also, if we have open sets U_i indexed in a set I we will denote their intersections by $U_{ij} := U_i \cap U_j$.

Definition. A presheaf F over X is said to be a *sheaf*, if for any covering $\{U_i\}_{i\in I}$ the diagram below is an equalizer.

$$F(U) \xrightarrow{e} \prod_{i\in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{i,j\in I} F(U_{ij}) \tag{3}$$

Where $\pi_i \circ e = F(U_i \subseteq U)$, $\pi_{i,j} \circ p = F(U_{ij} \subseteq U_i)$ and $\pi_{i,j} \circ q = F(U_{i,j} \subseteq U_j)$. And we write $\mathrm{Shv}(X)$ for the full subcategory of $\mathrm{PSh}(X)$ whose objects are sheaves on X.

Let S be any set, and $D \subseteq X$ be any irreducible closed set, then define the presheaf S^D (called the "skyscraper" of S at D) by:

$$S^D(U) = \begin{cases} S & \text{if } D \cap U \neq \emptyset \\ \{*\} & \text{otherwise} \end{cases}$$

And the restriction maps $S^D(U) \to S^D(V)$ are the identity if $D \cap V \neq \emptyset$ or $D \cap U = \emptyset$, and the unique map $S \to \{*\}$ if $D \cap U \neq \emptyset$ and $D \cap V = \emptyset$.

Proposition 10. *The skyscraper presheaf S^D is a sheaf if and only if $\boldsymbol{D}$ is an irreducible closed set.*

Proof. (sketch) the key to this proposition is noting that the subcategory of open sets which intersect D is a cofiltered category if and only if D is irreducible. □

Proposition 11. *For any topological space X, and any open subset $U \subseteq X$ the representable presheaf h_U is a sheaf on X.*

Proof. For any $V \in O(X)$, either $V \subseteq U$, in which case $h_U(V) = \{*\}$ or $h_U(V) = \emptyset$. In the first case, for any covering $\{V_i\}_{i\in I}$ we have $V_i \subseteq U$ and $V_i \cap V_j \subseteq U$, hence the diagram (3) becomes trivial with all components isomorphic to $\{*\}$. Otherwise, if $V \nsubseteq U$ we have $V_i \nsubseteq U$ for some $i \in I$, this necessarily yields that both $h_U(V)$ and $\prod h_U(V_i)$ are isomorphic to $\emptyset$ and hence $p = q$ and e is an isomorphism. □

Note also that for all open sets $U \subseteq X$, $h_X(U) = \{*\}$ and so h_X is terminal in $\mathrm{Shv}(X)$ and $\mathrm{PSh}(X)$.
Now define the stalk of a presheaf F at an irreducible closed set D as:

$$F_D := \operatorname*{Colim}_{U \cap D \neq \emptyset} F(U)$$

Note again that since the subcategory of open subsets of X intersecting D is cofiltered, this is a filtered colimit and thus a left exact functor by Theorem 7.

Theorem 12. *For any irreducible closed set $D \subseteq X$ the functor of stalks at $\boldsymbol{\mathcal{D}}$, $F \mapsto F_D$ is left adjoint to the skyscraper sheaf functor $S \mapsto S^D$.*

Proof. (idea) All one needs to do is recall how colimits are left adjoint to the diagonal functor. □

3. Grothendieck Topoi

3.1. Grothendieck Topologies.

Definition. Given a small category $\boldsymbol{\mathcal{C}}$, a *sieve*, S on some element $X \in \boldsymbol{\mathcal{C}}$ as a set of morphisms with codomain X, chosen to satisfy the condition that, if $f \in S$ then $f \circ g \in S$ for any g in $\boldsymbol{\mathcal{C}}$ so that the composition is defined.
A family of morphisms in $\boldsymbol{\mathcal{C}}$, $\{f_i : X_i \to X | i \in I\}$ *generates* S if

$$S = \bigcup_{i \in I} \{f_i \circ g \mid \mathsf{d}_1 g = X_i\}$$

Note that there a bijection between sieves S over X and subfunctors of the representable functor $h_X = \mathrm{Mor}_{\boldsymbol{\mathcal{C}}}(-, X)$.

Definition. Given a set of morphisms S with a common codomain X and a morphism $g : Y \to X$ we define the *pullback* of S along g as:

$$g^*(S) := \{f : Z \to Y \mid f \circ g \in S, Z \in \boldsymbol{\mathcal{C}}\}$$

Note that if S is a sieve on X, so is $g^*(S)$.

Definition. A *Grothendieck topology* on a category $\boldsymbol{\mathcal{C}}$ is a map J that assigns to each object $X \in \boldsymbol{\mathcal{C}}$ a collection of *covering sieves* $J(X)$ which are sieves on X satisfying:

(1) For all $X \in \boldsymbol{\mathcal{C}}$, $h_X \in J(X)$.
(2) If S is a sieve in $J(X)$ and $h : Y \to X$ is an arrow in $\boldsymbol{\mathcal{C}}$, then $h^*(S) \in J(Y)$.
(3) If $S \in J(X)$ and R is a sieve on X, so that for every arrow $h : Y \to X \in S$, $h^*(R) \in J(Y)$, then $R \in J(X)$.

Alternatively we can define a pretopology or a base for a Grothendieck topology:

Definition. A *Grothendieck pretopology* on a category $\boldsymbol{\mathcal{C}}$, or a *Grothendieck coverage*[1] is a map J' where for each $X \in \boldsymbol{\mathcal{C}}$, $J'(X)$ is a collection of *covering families* of X, which are families of morphisms $\{f_i : X_i \to X \mid i \in I\}$ satisfying:

(1) If $\phi : X' \to X$ is an isomorphism then $\{\phi\} \in J'(X)$.
(2) If $\{f_i : X_i \to X\} \in J'(X)$ then for any $g : Y \to X$ in $\boldsymbol{\mathcal{C}}$ there exists $\{h_l : Y_l \to Y\} \in J'(Y)$ such that each of the $g \circ h_j$ factor through some f_i.
(3) If $\{f_i : X_i \to X \mid i \in I\} \in J'(X)$ and $\{f_{i,j} : X_{i,j} \to X_i \mid j \in I_i\} \in J'(X_i)$ for each $i \in I$ then $\{f_i \circ f_{i,j} \mid i, \in I, j \in I_i\} \in J'(X)$.

Proposition 13. *Let J' be a pretopology, then the map J that, to each $X \in \boldsymbol{\mathcal{C}}$, assigns the family of sieves on X which contain some covering family in $J'(X)$ is a Grothendieck topology on $\boldsymbol{\mathcal{C}}$. J is the* topology generated by the pretopology *J'.*

[1]The more general notion of coverage admits only the second axiom.

A pair $(\boldsymbol{\mathcal{C}}, J)$ of a category with a topology on it is called a *site*. We often omit the topology where it is implied by the context.
Now note that given a site $(\boldsymbol{\mathcal{C}}, J)$ and any object $X \in \boldsymbol{\mathcal{C}}$, $J(X)$ has a natural structure of category: It is the subcategory of subfunctors of h_X with morphisms the inclusion morphisms.

Lemma 14. *For any object $X \in \boldsymbol{\mathcal{C}}$ and J a topology on $\boldsymbol{\mathcal{C}}$, both $J(X)$ and $J(X)^{op}$ are filtered categories.*

3.2. **The Zariski and Étale Sites.** Given a set X and a family of functions $\{X_i \xrightarrow{p_i} X \mid i \in I\}$ we say that this family is a *surjective family of functions* if $X = \bigcup_{i\in I} p_i(X_i)$.

Definition. Given a topological space X, the *Zariski site* $X_{\mathbf{Zar}}$ on X is the site where:

- The underlying category is $O(X)$.
- The pretopology is comprised of jointly surjective families of inclusions $\{U_i \subseteq U \mid i \in I\}$.

Note that to give a surjective family of inclusions of open sets into an open set U, is the same as to give an open cover of U.

Definition. Given a topological space X, we define the *étale site* $X_{\mathbf{\acute{E}t}}$ on X where:

- The underlying category, $\mathbf{\acute{E}t}(X)$.
- The topology is generated by surjective families of étale morphisms.

3.3. **Sheaves on a Site.**

Definition. Let $\boldsymbol{\mathcal{C}}$ be a category and J a topology on $\boldsymbol{\mathcal{C}}$, we say that a presheaf F on $\boldsymbol{\mathcal{C}}$ is a sheaf, if for every $x \in \boldsymbol{\mathcal{C}}$ and every sieve $S \in J(x)$ we have:

$$F(x) \cong \operatorname*{Lim}_{f\in S} F(\mathsf{d}_0 f) \tag{4}$$

If we assume that $\boldsymbol{\mathcal{C}}$ has pullbacks then we can reformulate the sheaf condition becomes instead of equation(4), the following diagram is an equalizer for any covering family $\{f_i : x_i \to x \mid i \in I\}$ (where the maps are analogous to those on equation (3)):

$$F(x) \xrightarrow{e} \prod_{i\in I} F(x_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j\in I} F(x_i \times_x x_j) \tag{5}$$

Note that this implies that the notions of a sheaf on the topological space X and a sheaf on $X_{\mathbf{Zar}}$ coincide.
Again we denote $\mathrm{PSh}(\boldsymbol{\mathcal{C}})$ the category of presheaves on $\boldsymbol{\mathcal{C}}$ and $\mathrm{Shv}(\boldsymbol{\mathcal{C}})$ the category of presheaves on the site $(\boldsymbol{\mathcal{C}}, J)$ (where J is implied).
A site be *subcanonical* if all representable presheaves are sheaves.

Proposition 15. *The Zariski and étale sites associated to a topological space X are subcanonical.*

Proof. The fact that the Zariski site is subcanonical was proved in Proposition 11. And the case for $X_{\mathbf{\acute{E}t}}$ is just a simple verification. □

3.4. **Sheafification of Presheaves.**

Definition. A presheaf $F \in \mathrm{PSh}(\boldsymbol{\mathcal{C}})$ is said to be a *separated presheaf* if the map from (4) is a monomorphism

Now, given a presheaf $F \in \mathrm{PSh}(\boldsymbol{\mathcal{C}})$ and a sieve R over $X \in \boldsymbol{\mathcal{C}}$ define:

$$F(X)_R := \operatorname*{Lim}_{U\to X\in R} F(U)$$

Where R is seen as a subcategory of $\boldsymbol{\mathcal{C}} \downarrow X$, note that an element of $F(X)_R$ corresponds to an R-compatible family of sections of F. With this said we define F^+ as:

$$F^+(X) := \operatorname*{Colim}_{R\in J(X)^{\mathrm{op}}} F(X)_R$$

Which is a presheaf since for any morphism $g : Y \to X$ and any sieve $R \in J(X)$ we have an obvious map $F(X)_R \to F(Y)_{g^*(R)}$.

Lemma 16. *For any presheaf F on the site $(\mathcal{C}, J)$ the presheaf F^+ defined above, is separated. Furthermore if F is separated, then F^+ is a sheaf.*

Proof. A proof of these statements can be found in [Joh02b]. ☐

Theorem 17. *Given any presheaf F the presheaf $F^{\#} := F^{++}$ is a sheaf, and furthermore $\#$ is a left exact left adjoint to the inclusion $i_{Shv(\mathcal{C})} : Shv(\mathcal{C}) \to PSh(\mathcal{C})$*

Proof. The functor $F \mapsto F^+$ is left exact since it is a filtered colimit (by Lemma 14), so necessarily $F \mapsto F^{\#} = F^{++}$ is left exact. For the adjointness property we refer back to the thesis. ☐

For the case of sheaves on a topological space we have a direct description of the sheafification process using the following theorem:

Theorem 18. *The categories $\mathbf{\acute{E}t}(X)$ and $Shv(X)$ are equivalent.*

Proof. The proof of this theorem can be seen in [Moe92], and an adaptation of it can be found in the thesis. ☐

3.5. **The Zariski and Étale Topoi.** Note that the inclusion of an open set is an étale map, and thus we have a canonical functor $i : O(X) \to \mathbf{\acute{E}t}(X)$. Furthermore if F is an étale sheaf then $F \circ i$ is a Zariski sheaf, and thus we have a functor:

$$i^* : \mathrm{Shv}(X_{\mathbf{\acute{E}t}}) \to \mathrm{Shv}(X_{\mathbf{Zar}})$$

The left Kan extension of $F : O(X)^{\mathrm{op}} \to \boldsymbol{Set}$ along $i^{\mathrm{op}} : O(X)^{\mathrm{op}} \to \mathbf{\acute{E}t}(X)^{\mathrm{op}}$ can be computed on an étale space $p : Y \to X$ by calculating the colimit over the comma category $(O(X)^{\mathrm{op}} \downarrow Y)$. The elements of this category are maps $f : U \to Y$ in $\mathbf{\acute{E}t}(X)^{\mathrm{op}}$ so that the diagram below commutes in $\mathbf{\acute{E}t}(X)^{\mathrm{op}}$:

(6)
$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Since we are doing this in category opposite to $\mathbf{\acute{E}t}(X)$, f is an étale map $\hat{f} : Y \to U$. Substituting f by $\hat{f}$ in diagram (6) we get $f(Y) \subseteq U$ and $f(Y) = p(Y)$, whence $p(Y) \subseteq U$. Thus the formula for the Kan extension satisfies:

$$i^K(F)(Y) = \underset{U \in (O(X)^{\mathrm{op}} \downarrow Y)}{\mathrm{Colim}} F(U) = \underset{U \supseteq p(Y)}{\mathrm{Colim}} F(U) \cong F(p(Y))$$

However, in general this is only a presheaf. But since the sheafification is left adjoint to the inclusion we get a left adjoint to i^*:

$$i_* := \# \circ i^K : \mathrm{Shv}(X_{\mathbf{Zar}}) \to \mathrm{Shv}(X_{\mathbf{\acute{E}t}})$$

Note also that i_* is left exact because it is a composition of left exact functors, and so we get:

Theorem 19. *The inclusion map $i : O(X) \to \mathbf{\acute{E}t}(X)$ induces an adjunction:*

$$i_* \dashv i^* : Shv(X_{\mathbf{Zar}}) \rightleftarrows Shv(X_{\mathbf{\acute{E}t}})$$

Where i_* is left exact.

Theorem 20. *Any sheaf $F \in Shv(X)$ is a colimit of representable sheaves.*

Proof. This is a direct consequence of the fact that every étale space is can be obtained as a colimit of open subsets of X and Theorem 18. ☐

Proposition 21. *Let $F, G \in Shv(X_{\mathbf{\acute{E}t}})$ and $\varphi : F \to G$ be a morphism so that for all $U \subseteq X$, $\varphi_U : F(U) \to G(U)$ is an isomorphism, then φ is an isomorphism.*

Proof. (idea) We only need to check the coverings of any étale space by lifts of open sets in X, since the φ_U are isomorphisms for all these sets, then this induces an isomorphism on the sheaf diagrams, and thus an isomorphism on the equalizers. □

Now recall the right Kan extension of a sheaf F on X to the étale site is given by:

$$i_K(F)(Y) := \operatorname*{Lim}_{f\in(Y\downarrow O(X)^{\mathrm{op}})} F(\mathsf{d}_1 f) = \operatorname*{Lim}_{\hat{f}\in(O(X)\downarrow Y)} F(\mathsf{d}_0\hat{f}) \tag{7}$$

Proposition 22. *If F a sheaf over the Zariski site, then its right Kan extension to the étale site is a sheaf $i_K(F) \in Shv(X_{\mathbf{\acute{E}t}})$.*

Proof. We prove this result in three steps: First we see that it has to satisfy the sheaf condition by coverings consisting solely of lifts of open sets of X, then we see that it has to satisfy for arbitrary open coverings of an étale space, and finally, since étale maps are open any étale cover factors uniquely through an open cover. □

Note now that for any open set U and any Zariski sheaf F $i_K(F)(U) = F(U)$ and thus by Proposition 21 we obtain that both the counit and the unit are isomorphisms, and thus, the following theorem holds:

Theorem 23. *The restriction functor $i^* : Shv(X_{\mathbf{\acute{E}t}}) \to Shv(X_{\mathbf{Zar}})$ is an equivalence of categories.*

This is an instance of the "comparison lemma" (see [Joh02b] Theorem 2.2.3 in section C2.2).

4. Geometric Morphisms and Point Functors

4.1. **Geometric Morphisms.** Grothendieck topoi are generalizations of categories of sheaves on topological spaces. So the definition of geometric morphism of topoi is chosen to mimic that of a continuous function of spaces. Here we just give the following theorem as motivation.

Definition. Given two topoi $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}$, a *geometric morphism* $f : \boldsymbol{\mathcal{E}} \to \boldsymbol{\mathcal{F}}$ is an adjoint pair $f^* \dashv f_* : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{E}}$ (where $f^* : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{E}}$) so that f^* is left exact.

Theorem 24. *If Y is Hausdorff then there is a bijection, up to natural isomorphism between geometric morphisms $f : Shv(X) \to Shv(Y)$ and continuous functions $f : X \to Y$.*

Definition. For a topos $\boldsymbol{\mathcal{E}}$ a functor $x^* : \boldsymbol{\mathcal{E}} \to \boldsymbol{Set}$ is said to be a *point functor* of $\boldsymbol{\mathcal{E}}$ if x^* is a left exact left adjoint.

Here we disregard the right adjoint since this introduces no further structure and is determined up to isomorphism, so equivalently we can say a point of $\boldsymbol{\mathcal{E}}$ is a geometric morphism $x : \boldsymbol{Set} \to \boldsymbol{\mathcal{E}}$.

4.2. **Point Functors of the Zariski Site.** Throughout this section x^* will be a point functor of a Zariski topos. Given any sheaf $F \in \mathrm{Shv}(X)$, and any open set $U \subseteq X$, the Yoneda lemma states that.

$$\mathrm{Mor}_{\mathrm{Shv}(X)}(h_U, F) = \mathrm{Nat}(\mathrm{Mor}_{O(X)}(-, U), F) \cong F(U)$$

Note also that since point functors are left adjoints, they are cocontinuous and since every sheaf $F \in \mathrm{Shv}(X)$ is a colimit of representables, x^* is completely determined by it's value on representables.
For any $V \subseteq X$ there is a unique arrow $h_V \to h_X$ which is a monomorphism. Since x^* is left exact, this gives a mono $x^*(h_V) \to x^*(h_X)$. Also since h_X is the terminal sheaf, $x^*(h_X)$ is terminal and so is isomorphic to $\{*\}$.
Now denote by $\mathbf{2} = \{0, 1\}$. Any $V \subseteq X$ induces a surjection:

$$\mathbf{2} \cong \mathrm{Mor}_{\boldsymbol{Set}}(\{*\}, \mathbf{2}) \cong \mathrm{Mor}_{\boldsymbol{Set}}(x^*(h_X), \mathbf{2}) \to \mathrm{Mor}_{\boldsymbol{Set}}(x^*(h_V), \mathbf{2}) \tag{8}$$

So let x_* be the right adjoint to x^*. Applying the isomorphism from the adjunction and the Yoneda lemma to equation (8) we get:

$$\mathrm{Mor}_{\boldsymbol{Set}}(x^*(h_V), \mathbf{2}) \cong \mathrm{Mor}_{\mathrm{Shv}(X)}(h_V, x_*(\mathbf{2})) \cong x_*(\mathbf{2})(V)$$

And hence there is a surjection $\mathbf{2} \to x_*(\mathbf{2})(V)$. Thus, for any open set $V \subseteq X$ and any point functor x^*:

$$x_*(\mathbf{2})(V) = \begin{cases} \text{either} & \mathbf{2} \\ \text{or} & \{*\} \end{cases}$$

Now it's easy to check that $x_*(\mathbf{2})$ must be a skyscraper of the set $\mathbf{2}$ on the complement of the open set where $x_*(\mathbf{2})$ is $\{*\}$. And since $x_*(\mathbf{2})$ is a sheaf this set must be an irreducible closed which we call D.
Now for any set S, let $\mathscr{P}(S)$ be the power set of S, and note that $S \cong S' = \{\{s\} : s \in S\} \subseteq \mathscr{P}(S)$. So let $\chi_{S'} : \mathscr{P}(S) \to \mathbf{2}$ to be the characteristic function of S' on $\mathscr{P}(S)$ ($\chi_{S'}(x) = 0$ if $x \subseteq S, |x| = 1$ and $\chi_{S'}(x) = 1$ otherwise) and 0 to be the constant function $\mathscr{P}(S) \to \mathbf{2}$ equal to 0 to obtain:

$$S \cong \mathsf{eq}\left(\mathscr{P}(S) \overset{\chi_{S'}}{\underset{0}{\rightrightarrows}} \mathbf{2}\right)$$

Noting also that $\mathscr{P}(S) \cong \mathrm{Mor}_{\boldsymbol{Set}}(S, \mathbf{2}) \cong \prod_{s \in S} \mathbf{2}$ and that x_* commutes with arbitrary limits:

$$x_*(S) \cong x_*\left(\mathsf{eq}\left(\prod_{s\in S} \mathbf{2} \rightrightarrows \mathbf{2}\right)\right) \cong \mathsf{eq}\left(\prod_{s\in S} x_*(\mathbf{2}) \rightrightarrows x_*(\mathbf{2})\right) \cong \mathsf{eq}\left(\prod_{s\in S} \mathbf{2}^D \rightrightarrows \mathbf{2}^D\right) \tag{9}$$

Since the product of skyscrapers over some (fixed) irreducible closed set D forms a skyscraper over D we get:

$$x_*(S)(U) \cong \begin{cases} \mathsf{eq}\left(\prod_{s\in S} \mathbf{2} \rightrightarrows \mathbf{2}\right) & \text{if } D \cap U \neq \emptyset \\ \{*\} & \text{otherwise} \end{cases}$$

And so:

$$x_*(S) \cong S^D$$

By uniqueness of the adjoint, the associated point functor x^* is naturally isomorphic to the functor of stalks $F \mapsto F_D$. And so we get:

Theorem 25. *Any point functor $x^* : Shv(X_{\mathbf{Zar}}) \to \boldsymbol{Set}$ is isomorphic to a functor of stalks over some irreducible closed set $D \subseteq X$.*

Now let $\mathrm{Pt}(X_{\mathbf{Zar}})$ be the full subcategory of $\mathrm{Func}(\mathrm{Shv}(X_{\mathbf{Zar}}), \boldsymbol{Set})$ whose objects are point functors and arrows are natural transformations. And let $\mathrm{Irr}(X)$ be the category of irreducible closed subsets of X.
Note that if $D \subseteq D'$ and $U \cap D \neq \emptyset$ then $U \cap D' \neq \emptyset$ so we have a map $F_D \to F_{D'}$ which is natural in F.
Let $\mathrm{St} : \mathrm{Irr}(X) \to \mathrm{Pt}(X_{\mathbf{Zar}})$ be the functor defined as:

$$\mathrm{St}(D)(F) := F_D$$

Proposition 26. *The functor $St : Irr(X) \to Pt(X_{\mathbf{Zar}})$ is an equivalence of categories.*

Proof. Theorem 25 guarantees that St is essentially surjective. Since the domain of St is a poset category it is also faithful. It remains to prove that it is injective on objects and full.
Let D, D' be two distinct irreducible closed sets of X so that $D \not\subseteq D'$, and let $V = X \setminus D'$. Then:

$$(h_V)_D \cong \{*\}$$
$$(h_V)_{D'} = \emptyset$$

Hence we conclude that $\mathrm{St}(D) \neq \mathrm{St}(D')$ and hence that St is injective on objects. Moreover under the above hypothesis that there can be no natural transformation $\mathrm{St}(D) \to \mathrm{St}(D')$, i.e there can only be a natural transformation $\mathrm{St}(D) \to \mathrm{St}(D')$ if $D \subseteq D'$.
Finally since any sheaf $F \in \mathrm{Shv}(X_{\mathbf{Zar}})$ is a colimit of representable sheaves $F = \operatorname{Colim}_i F_i$ and stalk functors are cocontinuous, a map $F_D \to F_{D'}$ is determined by a natural transformation $(F_i)_D \to (F_i)_{D'}$. These natural transformations are unique since $F_i \cong h_{U_i}$ and the maps $(h_{U_i})_D \to (h_{U_i})_{D'}$ are uniquely determined by U_i, D and D'. $\square$

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